LOSS RATE FOR A GENERAL LÉVY PROCESS WITH DOWNWARD PERIODIC BARRIER.

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Abstract

In this paper we consider a general Lévy process X reflected at downward periodic barrier A_t and constant upper barrier K giving a process $V_t^K = X_t + L_t^A - L_t^K$. We find the expression for a loss rate defined by $l^K = \mathbb{E}L_1^K$ and identify its asymptotics as $K \to \infty$ when X has light-tailed jumps and $\mathbb{E}X_1 < 0$.

Keywords: loss rate, Lévy process, barrier, queueing process.

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1. Introduction

In this paper we consider a general Lévy process X reflected at downward periodic barrier $A_t = \varphi(t + U)$ (for a periodic nonnegative function $\varphi(t)$ with period length s and U having uniform distribution on $[0, s]$) and at the constant upper barrier K, giving a process

$$V_t^K = X_t + L_t^A - L_t^K \tag{1}$$

being solution of respective Skorohod problem on each period where reflection is within bounded and convex set (see [5, 15]). In above we assume that $\varphi(t) \in [0, a]$ for some $a < K$. Process X is defined on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with the natural filtration that satisfies the usual assumptions of right continuity and completion. From now on we will also assume that jump measure ν of X is nonlattice.

In this paper we find the expression for a loss rate defined by

$$l^K = \mathbb{E}L_1^K, \tag{2}$$

where $\mathbb{E}$ denotes the expectation when reflected process is stationary with stationary measure π_K (that is $\mathbb{E}(\cdot) = \int_0^\infty \mathbb{E}[\cdot|V_0^K = x]\,\pi_K(dx)$), and prove that $l^K \sim De^{-\gamma K}$ as $K \to \infty$, where γ solves $\kappa(\gamma) = 0$ for a Laplace exponent $\kappa(\alpha) = \log \mathbb{E}\exp\{\alpha X(1)\}$ (which is well-defined in some set Θ) when X has light-tailed jumps and $\mathbb{E}X_1 < 0$.

The motivation comes from various queueing and telecommunication models (see [1, 3, 4, 6, 8, 14, 16]). Applications, where such a reflected Lévy process considered in this paper is natural, are models where additionally to the input and output mechanism modeled by a Lévy process there is a constant input given by a downward barrier A_t. This additional input is not available on liquid basis, but can only be used after some

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maturity date has been reached. We choose that this time lag is fixed and equals the length s of the period of φ (for exponential time delay see [9]). For example, in view of internet networking applications, one considers the combined behavior of two services (e.g. streaming video and some other data). The first input behaves like a Lévy process. The other input grows deterministically and can be served only at some fixed time s. The combined workload now behaves like a Lévy process reflected at a lower barrier A_t.

Fluid models with finite buffers are useful to model systems where losses are of crucial importance, as in inventory theory and telecommunications. Indeed, during the recent years real-time applications such as video streaming, interactive games have become increasingly popular among users. These applications are generally delay sensitive and require some preferential treatment in order to satisfy a desired level of Quality of Service. Traditionally finite capacity buffer mechanisms have been employed in the network routers, in which arriving packets are dropped when workload reaches its maximum capacity. In this paper we analyze the intensity of packets loss given by so-called loss rate given in (2).

There has been a great deal of work on overflow probabilities in various fluid and queueing models but there have been relatively few studies on the loss rate in finite buffer systems. When jumps of the Lévy process are heavy-tailed that one might have a hope to find relationship between these two notions (see [3, 6, 10] for more classical models). Still in our model we focus on light-tailed case and therefore we choose Kella-Whitt martingale approach [8]. In fact we follow the ideas included in seminal paper of Asmussen and Pihlsgård [2], where both barriers are constant (see also [13] for matrix analytic method). This case corresponds to the assumption that $A(t) \equiv 0$. Denoting the loss rate by $l^{K,0}$ analyzed in [2] we of course immediately get bounds:

$$l^{K,0} \le l^K \le l^{K-a,0}, \tag{3}$$

from which and [2, Th. 4.1] it follows e.g. that $\frac{1}{K} \log l^K = -\gamma$. In this paper we focus on more precise exact asymptotics.

The paper is organized as follows. In Section 2 we give preliminary results and in Section 3 the main results with proofs.

2. Preliminaries

Assume from now on that $\varphi \in \mathcal{C}^1(\operatorname{int} J_k)$ is invertible on some disjoint intervals J_k satisfying $\cup_{k=1}^n J_k = [0, s]$ with $\varphi'(x) \neq 0$ for $x \in \operatorname{int} J_k$.

Lemma 2.1. *Process* $A_t = \varphi(t + U)$ *has invariant measure:*

$$\xi(dy) = \sum_{k=1}^n \frac{1}{s} \left| h_k'(y) \right| \mathbf{1}_{\varphi(\operatorname{int} J_k)}(y) \, dy,$$

where h_k *is a inverse of* φ *on* $\operatorname{int} J_k$.

Proof. It is sufficient to check that for $t \in [0, s)$:

$$\mathbb{P}(A_t \le x) = \mathbb{P}(\varphi(t + U) \le x) = \frac{1}{s} \int_0^s \mathbf{1}_{(\varphi(t+u) \le x)} du = \frac{1}{s} \int_0^s \mathbf{1}_{(\varphi(u) \le x)} du , \tag{4}$$

where the last equality is a consequence of periodicity of φ. The second part of the theorem follows from a straightforward arguments concerning a distribution of a piecewise strictly monotone function of r.v.

$\square$

Example 2.1. The most interesting case for applications is a saw-like lower boundary modeling constant intensity input (with rate 1 for simplicity):

$$\varphi(t) = t \bmod a. \tag{5}$$

with $0 < a < K$. In this case $n = 1$, $J_1 = [0, a]$, $s = a$ and $\xi(dy) = \frac{dy}{a} \mathbf{1}_{(y \in [0, a])}$.

Example 2.2. More complex situation will appear when φ is composed from few lines with different slopes:

$$\varphi(t) = \begin{cases} t & \text{for } t \in [0, 1), \\ 1 - 2(t - 1) & \text{for } t \in [1, \frac{3}{2}), \\ 3\left(t - \frac{3}{2}\right) & \text{for } t \in [\frac{3}{2}, \frac{5}{2}). \end{cases} \tag{6}$$

In this case $n = 3$, $J_1 = [0, 1]$, $J_2 = [1, \frac{3}{2})$, $J_3 = [\frac{3}{2}, \frac{5}{2})$, $s = \frac{5}{2}$ and $\xi(dy) = \frac{2}{5}\left(\mathbf{1}_{\varphi(J_1)}(y) + \frac{1}{2}\mathbf{1}_{\varphi(J_2)}(y) + \frac{1}{3}\mathbf{1}_{\varphi(J_3)}(y)\right) dy$.

Following arguments of Asmussen [1, p. 393-394] we have the following representation of stationary distribution of V_t^K.

Lemma 2.2. *The stationary distribution V_∞^K of the two-sided reflected Lévy process is given by*

$$\mathbb{P}(V_\infty^K \geq x) = \int_0^a \underbrace{\sum_{k=1}^n \mathbb{P}(X_\tau \geq \widehat{A}_\tau^{-z,k} + x)p_k(z)}_{\overline{\pi}_K^z(x)}\,\xi(dz),$$

where $\tau = \inf\{t \geq 0 : X_t \notin [x - K, \widehat{A}_t^{-z,k} + x)\}$, *and* $\widehat{A}_t^{-z,k} = -\varphi(h_k(z) - t)$ *for* $z \in \varphi(J_k)$ *and* $p_k(z) = \mathbb{P}(U \in J_k | \varphi(U) = z) = \frac{|h_k'(z)|\mathbf{1}_{\varphi(J_k)}(z)}{\sum_{j=1}^n |h_j'(z)|\mathbf{1}_{\varphi(J_j)}(z)}$.

Lemma 2.3. *If* $\mathbb{E}|X_1| < \infty$ *then* $\mathbb{E}L_t^A < \infty$ *and* $\mathbb{E}L_t^K < \infty$ *for each* $t \geq 0$.

Proof. Note that by (1) we have $\mathbb{E}L_t^K < \infty$ if $\mathbb{E}L_t^A < \infty$. Condition $\mathbb{E}L_t^A < \infty$ follows from the Wald identity applied to the random walk with increments being corrections of X between consecutive visits of the downward barrier (for details see [2]).

$\square$

Further we need a little modification of Lévy exponent $\kappa(\alpha)$. We will treat large and small jumps separately. Let L be a constant that satisfies $L > \max(K, 1)$. Then $\kappa(\alpha)$ can be rewritten as follows:

$$\theta_L \alpha + \frac{\sigma^2 \alpha^2}{2} + \int_{-\infty}^\infty [e^{\alpha x} - 1 - \alpha x \mathbf{1}_{(|x| \leq L)}]\nu(dx), \quad \alpha \in \Theta, \tag{7}$$

where $\theta_L = \theta + \int_1^L x\nu(dx) + \int_{-L}^{-1} x\nu(dx)$.

For any process Y we will denote by $\{Y_t^c\}$ its continuous part and the jumps by $\Delta Y_s = Y_s - Y_{s-}$.

We split ΔL_t^K into two parts, $\underline{\Delta} L_t^K$ and $\overline{\Delta} L_t^K$, corresponding to $\Delta X_s \in [0, L]$ and $\Delta X_s \in (L, \infty)$, respectively, and ΔL_t^A into $\underline{\Delta} L_t^A$ and $\overline{\Delta} L_t^A$, corresponding to $\Delta X_s \in [-L, 0]$ and $\Delta X_s \in (-\infty, -L)$, respectively. Let $\underline{l}_j^K = \mathbb{E} \sum_{0 \le s \le 1} \underline{\Delta} L_s^K$, $\overline{l}_j^K = \mathbb{E} \sum_{0 \le s \le 1} \overline{\Delta} L_s^K$, $\underline{l}_j^A = \mathbb{E} \sum_{0 \le s \le 1} \underline{\Delta} L_s^A$, $\overline{l}_j^A = \mathbb{E} \sum_{0 \le s \le 1} \overline{\Delta} L_s^A$. Then $l_j^K = \underline{l}_j^K + \overline{l}_j^K$ and $l_j^A = \underline{l}_j^A + \overline{l}_j^A$. Finally, let $l_c^K = l^K - l_j^K$ and $l_c^A = l^A - l_j^A$ with $l^A = \mathbb{E} L_1^A$.

Theorem 2.1. *For $\alpha \in \Theta$,*

$$
\begin{aligned}
M_t = \quad & \kappa(\alpha) \int_0^t e^{\alpha V_s^K} ds + e^{\alpha V_0^K} - e^{\alpha V_t^K} + \alpha \int_0^t e^{\alpha A_s} dL_s^{A,c} \\
& + \sum_{0 \le s \le t} e^{\alpha A_s}(1 - e^{-\alpha \Delta L_s^A}) - \alpha e^{\alpha K} L_t^{K,c} + e^{\alpha K} \sum_{0 \le s \le t} (1 - e^{\alpha \Delta L_s^K}) \quad (8)
\end{aligned}
$$

is a zero-mean martingale.

Proof. It is well-known that for an $\mathcal{F}_t$-adapted process $Y_t = \int_0^t dY_s^c + \sum_{0 \le s \le t} \Delta Y_s$ of locally bounded variation, process

$$
K_t = \kappa(\alpha) \int_0^t e^{\alpha Z_s} ds + e^{\alpha x} - e^{\alpha Z_t} + \alpha \int_0^t e^{\alpha Z_s} dY_s^c + \sum_{0 \le s \le t} e^{\alpha Z_s}(1 - e^{-\alpha \Delta Y_s})
$$

is a local martingale whenever $\alpha \in \Theta$, where $Z_t = x + X_t + Y_t$. Taking $Y_t = L_t^A - L_t^K$ and using Lemma 2.3 to prove that Y has locally bounded variation, we get that

$$
\begin{aligned}
M_t = \quad & \kappa(\alpha) \int_0^t e^{\alpha V_s^K} ds + e^{\alpha V_0^K} - e^{\alpha V_t^K} \\
& + \alpha \int_0^t e^{\alpha V_s^K} dL_s^{A,c} + \sum_{0 \le s \le t} e^{\alpha V_s^K}(1 - e^{-\alpha \Delta L_s^A}) \\
& - \alpha \int_0^t e^{\alpha V_s^K} dL_s^{K,c} + \sum_{0 \le s \le t} e^{\alpha V_s^K}(1 - e^{\alpha \Delta L_s^K})
\end{aligned}
$$

is a local martingale. M_t equals (8) since $V_s^K = K$ just after a jump of L_s^K, and $V_s^K = A_s$ just after a jump of L_s^A. To prove that $\{M_t\}$ is a true martingale it sufficient to show that $\mathbb{E} \sup_{0 \le s \le t} M_s < \infty$. This follows from the following conditions: $V_t^K \le K$, $\mathbb{E} L_t^{A,c}$, $\mathbb{E} L_t^{K,c} < \infty$ and $\mathbb{E} \sum_{0 \le s \le t} |1 - e^{\alpha \Delta L_s^K}| < \infty$, $\mathbb{E} \sum_{0 \le s \le t} |1 - e^{-\alpha \Delta L_s^A}| < \infty$ (see also the proof of [2, Prop. 3.1]). $\qquad \square$

Corollary 2.1. *Let $\alpha \in \Theta$. Then l^K satisfies the following equation:*

$$
\begin{aligned}
\alpha(1 - e^{\alpha K})l^K &= -\kappa(\alpha)\mathbb{E}e^{\alpha V_0^K} + \alpha\mathbb{E}X_1 - \alpha e^{\alpha K}\bar{l}_j^K + \alpha\bar{l}_j^A \\
&\quad + \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\Delta L_s^K)^2 + \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\Delta L_s^A)^2 \\
&\quad - e^{\alpha K}\mathbb{E}\sum_{0\le s\le 1}(1 - e^{\alpha\overline{\Delta}L_s^K}) - \mathbb{E}\sum_{0\le s\le 1}e^{\alpha A_s}(1 - e^{-\alpha\overline{\Delta}L_s^A}) \\
&\quad - \alpha^2\mathbb{E}\int_0^1 A_s dL_s^{A,c} - \alpha^2\mathbb{E}\sum_{0\le s\le 1}A_s\Delta L_s^A + o(\alpha^2)
\end{aligned}
\tag{9}
$$

Proof. If we take $t = 1$ in M_t and use the stationarity of V_t^K we get

$$
\begin{aligned}
0 = \kappa(\alpha)\mathbb{E}e^{\alpha V_0^K} &+ \alpha\mathbb{E}\int_0^1 e^{\alpha A_s}dL_s^{A,c} + \mathbb{E}\sum_{0\le s\le 1}e^{\alpha A_s}(1 - e^{-\alpha\Delta L_s^A}) \\
&- \alpha e^{\alpha K}l_c^K + e^{\alpha K}\mathbb{E}\sum_{0\le s\le 1}(1 - e^{\alpha\Delta L_s^K}).
\end{aligned}
\tag{10}
$$

Moreover, we have

$$
\sum_{0\le s\le 1}(1 - e^{\alpha\Delta L_s^K}) = \sum_{0\le s\le 1}(1 - e^{\alpha\underline{\Delta}L_s^K}) + \sum_{0\le s\le 1}(1 - e^{\alpha\overline{\Delta}L_s^K}),
\tag{11}
$$

$$
\sum_{0\le s\le 1}e^{\alpha A_s}(1 - e^{-\alpha\Delta L_s^A}) = \sum_{0\le s\le 1}e^{\alpha A_s}(1 - e^{-\alpha\underline{\Delta}L_s^A}) + \sum_{0\le s\le 1}e^{\alpha A_s}(1 - e^{-\alpha\overline{\Delta}L_s^A}).
\tag{12}
$$

Applying expansion:

$$
e^{\alpha x} = 1 + \alpha x + \frac{(\alpha x)^2}{2} + \frac{(\alpha x)^3}{6}e^{\theta\alpha x}, \quad \theta \in [-1, 1]
\tag{13}
$$

to the first parts of the right-hand sides of (11) and (12) and expansion $e^\alpha = 1 + \alpha + o(\alpha)$ to $\alpha\mathbb{E}\int_0^1 e^{\alpha A_s}dL_s^{A,c}$ completes the proof.

$\square$

We will need also the following observation.

Lemma 2.4. *We have:*

$$
\mathbb{E}\int_0^1 A_s dL_s^A = \mathbb{E}(A_0)l^A,
$$

where $\mathbb{E}A_0 = \int_0^a y\,\xi(dy)$.

Proof. Note that

$$
\mathbb{E}L_s^A = s\mathbb{E}L_1^A = sl^A
$$

which is consequence of the fact that L_s^A has independent and stationary increments under invariant starting position of A. The proof follows now from the following:

$$
\begin{aligned}
\mathbb{E} \int_0^1 A_s dL_s^A &= \int_0^a \mathbb{E} \int_0^1 A_s^z dL_s^A \xi(dz) \\
&= \int_0^a \left[\mathbb{E} A_s^z L_s^A \big|_0^1 - \mathbb{E} \int_0^1 L_s^A dA_s^z \right] \xi(dz) \\
&= \int_0^a \left[A_1^z l^A - \int_0^1 s l^A dA_s^z \right] \xi(dz) \\
&= \int_0^a \left[A_1^z l^A - l^A (s A_s^z \big|_0^1 - \int_0^1 A_s^z ds) \right] \xi(dz) \\
&= \int_0^a l^A \int_0^1 A_s^z ds \xi(dz) = l^A \int_0^1 \int_0^a A_s^z \xi(dz) ds = l^A \mathbb{E} A_0.
\end{aligned}
$$

□

Now using Lemma 2.4 we can rewrite (9) as follows.

Lemma 2.5. *As $\alpha \downarrow 0$ we have*

$$
\begin{aligned}
\alpha(1 - e^{\alpha K} + \alpha \mathbb{E}(A_0)) l^K &= -\kappa(\alpha) \mathbb{E} e^{\alpha V_0^K} + \alpha \mathbb{E} X_1 - \alpha e^{\alpha K} \bar{l}_j^K + \alpha \bar{l}_j^A \\
&\quad + \frac{\alpha^2}{2} \mathbb{E} \sum_{0 \le s \le 1} (\underline{\Delta} L_s^K)^2 + \frac{\alpha^2}{2} \mathbb{E} \sum_{0 \le s \le 1} (\underline{\Delta} L_s^A)^2 \\
&\quad - e^{\alpha K} \mathbb{E} \sum_{0 \le s \le 1} (1 - e^{\alpha \overline{\Delta} L_s^K}) - \mathbb{E} \sum_{0 \le s \le 1} e^{\alpha A_s} (1 - e^{-\alpha \overline{\Delta} L_s^A}) \\
&\quad + \alpha^2 \mathbb{E}(A_0) \mathbb{E} X_1 + \alpha^2 \mathbb{E} \sum_{0 \le s \le 1} A_s \overline{\Delta} L_s^A + o(\alpha^2). \qquad (14)
\end{aligned}
$$

3. Main results

The first main result gives representation of l^K in terms of basic characteristics of the process X and lower boundary A.

Theorem 3.1. *Let $\{X_t\}$ be a Lévy process and l^K the loss rate defined in (2). If $\int_1^\infty y\nu(dy) = \infty$, then $l^K = \infty$, and otherwise*

$$
\begin{aligned}
l^K &= \mathbb{E} X_1 \left[\frac{1}{K - \mathbb{E} A_0} \int_0^K x \pi_K(dx) - \frac{\mathbb{E} A_0}{(K - \mathbb{E} A_0)} \right] + \frac{\sigma^2}{2(K - \mathbb{E} A_0)} \qquad (15) \\
&\quad + \frac{1}{2(K - \mathbb{E} A_0)} \int_0^a \int_z^K \int_{-\infty}^\infty \varphi_K(x, y, z) \nu(dy) \pi_K^z(dx) \xi(dz),
\end{aligned}
$$

where

$$
\varphi_K(x, y, z) = \begin{cases} -(x - z)^2 - 2y(x - z) & \text{if } y \le -x + z, \\ y^2 & \text{if } -x + z < y < K - x, \\ 2y(K - x) - (K - x)^2 & \text{if } y \ge K - x. \end{cases}
$$

Proof. The first claim follows immediately if we note that for $\int_1^\infty y\nu(dy) = \infty$ and $L > K$ we have:

$$l^K \geq \int_0^K \pi_K(dx) \int_L^\infty (y - K + x)\nu(dy) \geq \int_L^\infty (y - K)\nu(dy) = \infty.$$

The idea of the proof of the second part of the theorem is based on two steps: **1)** expanding all terms on the right-hand side of (14) and **2)** sending L to infinity and then $\alpha \downarrow 0$.

Step 1. In particular, the first term on r.h.s. of (14) equals:

$$
\begin{aligned}
\kappa(\alpha)\mathbb{E}e^{\alpha V_0^K} = \quad & \int_0^K e^{\alpha x} \int_{-\infty}^\infty e^{\alpha y} \mathbf{1}_{(|y|\geq L)}\nu(dy)\pi_K(dx) \\
& - \int_{-\infty}^\infty \mathbf{1}_{(|y|\geq L)}\nu(dy) + \alpha \left(\theta_L - \int_0^K x \int_{-\infty}^\infty \mathbf{1}_{(|y|\geq L)}\nu(dy)\pi_K(dx) \right) \\
& + \alpha^2 \left(\theta_L \int_0^K x\pi_K(dx) + \frac{\sigma^2}{2} + \int_{-L}^L \frac{y^2}{2}\nu(dy) \right. \\
& \left. - \int_0^K \frac{x^2}{2} \int_{-\infty}^\infty \mathbf{1}_{(|y|\geq L)}\nu(dy)\pi_K(dx) \right) + o(\alpha^2).
\end{aligned}
$$

Similarly,

$$\alpha\mathbb{E}X_1 = \alpha\theta_L + \alpha \int_{-\infty}^\infty y\mathbf{1}_{(|y|\geq L)}\nu(dy),$$

$$\frac{\alpha^2}{2}\mathbb{E}\sum_{0\leq s\leq 1} (\underline{\Delta}L_s^K)^2 = \frac{\alpha^2}{2} \int_0^K \pi_K(dx) \int_{K-x}^L (y - K + x)^2\nu(dy)$$

and

$$\frac{\alpha^2}{2}\mathbb{E}\sum_{0\leq s\leq 1} (\underline{\Delta}L_s^A)^2 = \frac{\alpha^2}{2} \int_0^a \int_z^K \int_{-L}^{-x+z} (x + y - z)^2\nu(dy)\pi_K^z(dx)\xi(dz).$$

For $x > 0$ denote $\overline{\nu}(x) = \nu((x, \infty))$ and similarly for $x < 0$ let $\underline{\nu}(x) = \nu((-\infty, x))$. Then:

$$\alpha\overline{l}_j^A = -\alpha \int_{-\infty}^{-L} y\nu(dy) - \alpha \int_0^K x\pi_K(dx)\underline{\nu}(-L) + \alpha \int_0^a \int_z^K z\pi_K^z(dx)\xi(dz)\underline{\nu}(-L)$$

and

$$
\begin{aligned}
\alpha e^{\alpha K}\overline{l}_j^K = \quad & \alpha \int_L^\infty y\nu(dy) + \alpha^2 K \int_L^\infty y\nu(dy) \\
& + (\alpha + \alpha^2 K) \left[\int_0^K x\pi_K(dx)\overline{\nu}(L) - K\overline{\nu}(L) \right] + o(\alpha^2).
\end{aligned}
$$

The other terms equal:

$$e^{\alpha K}\mathbb{E}\sum_{0\leq s\leq 1}(1-e^{\alpha\overline{\Delta}L_s^K}) \;\;=\;\; (1+\alpha K+\frac{\alpha^2 K^2}{2})\overline{\nu}(L)-\int_0^K e^{\alpha x}\int_L^\infty e^{\alpha y}\nu(dy)\pi_K(dx)+o(\alpha^2)$$

and

$$\mathbb{E}\sum_{0\leq s\leq 1}e^{\alpha A_s}(1-e^{-\alpha\overline{\Delta}L_s^A}) \;\;=\;\; \mathbb{E}\sum_{0\leq s\leq 1}(1-e^{-\alpha\overline{\Delta}L_s^A})+\alpha^2\mathbb{E}\sum_{0\leq s\leq 1}A_s\overline{\Delta}L_s^A+o(\alpha^2)$$

with

$$\mathbb{E}\sum_{0\leq s\leq 1}(1-e^{-\alpha\overline{\Delta}L_s^A}) \;\;=\;\; \underline{\nu}(-L)-\int_0^K e^{\alpha x}\int_{-\infty}^{-L}e^{\alpha y}\nu(dy)\pi_K(dx)$$

$$+\int_0^a \alpha z\int_z^K e^{\alpha x}\int_{-\infty}^{-L}e^{\alpha y}\nu(dy)\pi_K^z(dx)\xi(dz)$$

$$-\frac{1}{2}\int_0^a \alpha^2 z^2\int_z^K e^{\alpha x}\int_{-\infty}^{-L}e^{\alpha y}\nu(dy)\pi_K^z(dx)\xi(dz)+o(\alpha^2).$$

Step 2. If we now rearrange all terms of (14) using above identities and let $L\to\infty$ (note that $\theta_L\to\mathbb{E}X_1$ as $L\to\infty$) we get:

$$\alpha(1-e^{\alpha K}+\alpha\mathbb{E}A_0)l^K = \quad -\mathbb{E}X_1\alpha^2\int_0^K x\pi_K(dx)-\frac{\sigma^2\alpha^2}{2}$$

$$-\frac{\alpha^2}{2}\int_0^a\int_z^K\int_{-x+z}^{K-x}y^2\nu(dy)\pi_K^z(dx)\xi(dz)$$

$$+\frac{\alpha^2}{2}\int_0^a\int_z^K\int_{K-x}^\infty((x-K)^2+2y(x-K))\nu(dy)\pi_K^z(dx)\xi(dz)$$

$$+\frac{\alpha^2}{2}\int_0^a\int_z^K\int_{-\infty}^{-x+z}((x-z)^2+2y(x-z))\nu(dy)\pi_K^z(dx)\xi(dz)$$

$$+\alpha^2\mathbb{E}A_0\mathbb{E}X_1+o(\alpha^2).$$

The proof is completed by dividing both sides of above equation by $\alpha(1-e^{\alpha K}+\alpha\mathbb{E}A_0)$ and sending α to 0.

$\square$

Assume now that there exists $\gamma>0$ ($\gamma\in\Theta$) such that $\kappa(\gamma)=0$. Define now the new probability measure:

$$\left.\frac{d\mathbb{P}^\gamma}{d\mathbb{P}}\right|_{\mathcal{F}_t}=e^{\gamma X_t}$$

for which we have $\mathbb{E}^\gamma X_1=\kappa^{'}(\gamma)>0$ since on $\mathbb{P}^\gamma$ process X is a Lévy process with Laplace exponent $\kappa_\gamma(\alpha)=\kappa(\alpha+\gamma)$. We also need two passage times:

$$\tau_z^A(x)=\inf\{t\geq 0:X_t\geq\widehat{A}_t^{-z}+x\},\qquad \tau_{-z}^-=\inf\{t\geq 0:X_t<-z\}.$$

Further, let $\tau^A(x)=\inf\{t\geq 0:X_t\geq\widehat{A}_t^\xi+x\}$, where $\widehat{A}_t^\xi=\int_0^\infty A_t^{-y}\xi(dy)$ and $B^A(x)=X_{\tau^A(x)}-x$. The second main result concerns the asymptotics of l^K as $K\to\infty$.

Theorem 3.2. *Assume that there exists $\gamma > 0$ ($\gamma \in \Theta$) such that $\kappa(\gamma) = 0$ and $\kappa'(\gamma) < \infty$. Then there exists random variable $B^A(\infty)$ such that:*

$$\lim_{x \to \infty} \mathbb{E}^\gamma e^{-\gamma B^A(x)} = \mathbb{E}^\gamma e^{-\gamma B^A(\infty)}. \tag{16}$$

Furthermore, there exists finite constant D such that:

$$l^K \sim D e^{-\gamma K}, \qquad \text{as } K \to \infty, \tag{17}$$

where we write $f(K) \sim g(K)$ when $\lim_{K \to \infty} f(K)/g(K) = 1$, and

$$
\begin{aligned}
D &= -\mathbb{E} X_1 C_\gamma + \mathbb{E}^\gamma e^{-\gamma B^A(\infty)} \int_0^\infty e^{\gamma x} \mathbb{P}^\gamma(\tau_{-x}^- = \infty) \int_x^\infty \left(1 - e^{\gamma(y-x)}\right) \nu(dy)\, dx \\
&\quad + \int_{-\infty}^0 \left(y + \gamma^{-1}\left(1 - e^{\gamma y}\right)\right) \nu(dy) \\
&\quad + \int_0^\infty \int_0^{a \wedge x} \mathbb{P}(\tau_z^A(x) < \infty) \int_{-\infty}^{-x+z} \left(1 - e^{\gamma(x+y-z)}\right) \nu(dy)\, \xi(dz)\, dx \tag{18}
\end{aligned}
$$

with

$$C_\gamma = \mathbb{E} e^{\gamma A_s}.$$

Proof. The proof is based on the following observation:

$$l^K = \frac{e^{\gamma K}}{e^{\gamma K} - C_\gamma} I_1 + \frac{1}{e^{\gamma K} - C_\gamma} I_2 + \frac{e^{\gamma K} \gamma^{-1}}{e^{\gamma K} - C_\gamma} I_3 + \frac{\gamma^{-1}}{e^{\gamma K} - C_\gamma} I_4 - \frac{C_\gamma \mathbb{E} X_1}{e^{\alpha K} - C_\gamma}, \tag{19}$$

where

$$I_1 = \int_0^K \int_{K-x}^\infty (y - K + x)\nu(dy)\pi_K(dx),$$

$$I_2 = \int_0^a \int_0^K \int_{-\infty}^{-x+z} (x + y - z)\nu(dy)\pi_K^z(dx)\xi(dz),$$

$$I_3 = \int_0^K \int_{K-x}^\infty (1 - e^{\gamma(y-K+x)})\nu(dy)\pi_K(dx),$$

$$I_4 = \int_0^a \int_0^K \int_{-\infty}^{-x+z} (1 - e^{\gamma(x+y-z)})\nu(dy)\pi_K^z(dx)\xi(dz).$$

Indeed, note that from equation (7) taking $\alpha = \gamma$ we get:

$$0 = \gamma C_\gamma l_c^A - \gamma e^{\gamma K} l_c^K + C_\gamma \mathbb{E} \sum_{0 \le s \le 1} (1 - e^{-\gamma \Delta L_s^A}) + e^{\gamma K} \mathbb{E} \sum_{0 \le s \le 1} (1 - e^{\gamma \Delta L_s^K}),$$

where we used fact that $\mathbb{E} \int_0^1 e^{\gamma A_s} dL_s^A = C_\gamma l_c^A$. Let $\epsilon > 0$. We split ΔL_t^K into two parts, $\underline{\Delta}^\epsilon L_t^K$ and $\overline{\Delta}^\epsilon L_t^K$, corresponding to $\Delta X_s \in [0, \epsilon]$ and $\Delta X_s \in (\epsilon, \infty)$, respectively, and ΔL_t^A into $\underline{\Delta}^\epsilon L_t^A$ and $\overline{\Delta}^\epsilon L_t^A$, corresponding to $\Delta X_s \in [-\epsilon, 0]$ and $\Delta X_s \in (-\infty, -\epsilon)$, respectively. Now we have:

$$e^{\gamma K} \mathbb{E} \sum_{0 \le s \le 1} (1 - e^{\gamma \underline{\Delta}^\epsilon L_s^K}) = e^{\gamma K} \left(-\gamma(l_j^K - \bar{l}_j^K) - \frac{\gamma^2}{2} \mathbb{E} \sum_{0 \le s \le 1} (\underline{\Delta}^\epsilon L_s^K)^2\right) + o(\epsilon^2),$$

$$\mathbb{E} \sum_{0 \leq s \leq 1} (1 - e^{-\gamma \underline{\Delta}^\epsilon L_s^A}) = \gamma(l_j^A - \bar{l}_j^A) - \frac{\gamma^2}{2} \mathbb{E} \sum_{0 \leq s \leq 1} (\underline{\Delta}^\epsilon L_s^A)^2 + o(\epsilon^2).$$

Thus,

$$
\begin{aligned}
0 &= \gamma C_\gamma l_c^A - \gamma e^{\gamma K} l_c^K + C_\gamma \mathbb{E} \sum_{0 \leq s \leq 1} (1 - e^{-\gamma \overline{\Delta}^\epsilon L_s^A}) + e^{\gamma K} \mathbb{E} \sum_{0 \leq s \leq 1} (1 - e^{\gamma \overline{\Delta}^\epsilon L_s^K}) \\
&+ \gamma C_\gamma (l_j^A - \bar{l}_j^A) - \frac{\gamma^2}{2} C_\gamma \mathbb{E} \sum_{0 \leq s \leq 1} (\underline{\Delta}^\epsilon L_s^A)^2 - e^{\gamma K} \gamma(l_j^K - \bar{l}_j^K) - \frac{\gamma^2}{2} \mathbb{E} \sum_{0 \leq s \leq 1} (\underline{\Delta}^\epsilon L_s^K)^2 \\
&+ o(\epsilon^2).
\end{aligned}
$$

Using the fact that $l^A = l^K - \mathbb{E}X_1$ we have:

$$
\begin{aligned}
l^K (C_\gamma - e^{\gamma K}) \gamma &= \gamma C_\gamma \mathbb{E}X_1 + \gamma C_\gamma \bar{l}_j^A - \gamma e^{\gamma K} \bar{l}_j^K - C_\gamma \mathbb{E} \sum_{0 \leq s \leq 1} (1 - e^{-\gamma \overline{\Delta}^\epsilon L_s^A}) \\
&- e^{\gamma K} \mathbb{E} \sum_{0 \leq s \leq 1} (1 - e^{\gamma \overline{\Delta}^\epsilon L_s^K}) + \frac{\gamma^2}{2} C_\gamma \mathbb{E} \sum_{0 \leq s \leq 1} (\underline{\Delta}^\epsilon L_s^A)^2 + \frac{\gamma^2}{2} \mathbb{E} \sum_{0 \leq s \leq 1} (\underline{\Delta}^\epsilon L_s^K)^2 \\
&+ o(\epsilon^2).
\end{aligned}
$$

If we send $\epsilon \to 0$ we get:

$$l^K (C_\gamma - e^{\gamma K}) \gamma = \gamma C_\gamma \mathbb{E}X_1 - \gamma C_\gamma I_2 - \gamma e^{\gamma K} I_1 - C_\gamma I_4 - e^{\gamma K} I_3.$$

Now (19) follows by dividing by $(C_\gamma - e^{\gamma K})\gamma$.

Note that I_1 and I_3 are the same like those in [2, Theorem 3.2] and I_2 and I_4 have just additional integral over $\xi(dz)$ and we should take $x - z$ instead of x under the integrals sign. Using thus the same arguments like in the proof of [2, Theorem 4.1] will now completes the proof when we prove weak convergence (16). To this one can use classical renewal arguments applied to the process $\{X_{\tau^A(an)}, n \in N\}$ (by considering *ladder height lines* being $\widehat{A}_t$ starting from invariant measure shifted by a from the previous position of ladder process).

$\square$

Example 3.1. For a stable $M/M/1$ queue, that is for $X_t = \sum_{i=1}^{N_t} \sigma_i - t$ with $\{\sigma_i\}_{\{i \geq 1\}}$ being i.i.d. exponentially r.v.'s with intensity μ and N_t being a Poisson process with intensity $\lambda < \mu$, we have $\nu(dx) = \mu \lambda e^{-\mu x} dx$ and $\mathbb{P}^\gamma(\tau_{-x}^- = \infty) = 1 - e^{-\gamma x}$, where $\gamma = \mu - \lambda$, since considering X on $\mathbb{P}^\gamma$ is equivalent of exchanging intensities of arrival and service processes. Moreover, choosing saw-like lower boundary given in (5) by lack of memory of exponential distribution on $\mathbb{P}^\gamma$ we have $B^A(\infty) = e_1 - Y$, where Y has uniform distribution $\xi(dx) = \frac{dx}{a}$ $(x \in [0, a])$ and e_1 is exponential r.v. with intensity λ. This gives:

$$D = \frac{1}{a} \left(e^{a(\mu - \lambda)} - 1 \right) \frac{\mu - \lambda}{\mu} \frac{\lambda}{\mu}.$$

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